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On the convergence of splitting proximal methods for equilibrium problems in Hilbert spaces

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ABSTRACT

Two splitting procedures for solving equilibrium problems involving the sum of two bifunctions are proposed and their convergence is established under mild assumptions.

1. Introduction and preliminaries

Throughout this paper, H is a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and denote by S the set of solutions to the following equilibrium problem:

$$\text{find } x \in C \quad \text{such that} \quad F(x, y) \geqslant 0, \quad \forall y \in C, \tag{1.1}$$

$F(x, y) := F_1(x, y) + F_2(x, y)$, where $F_1, F_2 : C \times C \to \mathbb{R}$ are two given bifunctions.

We will focus our attention on problem (1.1) which is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minmax problems, Nash equilibrium problem in noncooperative games and others (see, for instance [3,9] and the references quoted therein). In recent years, methods for solving equilibrium problems have been studied extensively, see for example [1,5,12]. Since, in general, it is difficult to evaluate the resolvent operator. One alternative is to decompose the given bifunction into the sum of two (or more) bifunctions whose resolvent are easier to evaluate than the resolvent of the original one. Such a method, in the context of variational inequalities, is known as a splitting method. This can lead to the development of very efficient methods, since one can treat each part of the original bifunction independently. In the context of variational inequalities splitting methods and related techniques have been studied by many authors. Of most interest are works by Passty [11], Lehdili and Lemaire [6], Moudafi and Théra [10] who proved ergodic convergence of the proximal method in the context of variational inequalities. Motivated by these works and that by Aoyama–Takahashi [1], we introduce two splitting methods in order to approximate a solution of (1.1). We will prove their convergence to some solution of the equilibrium problem in ergodic sense, we also provide some developments, and show that our convergence theorems improve, unify and develop several corresponding results in, for instance, variational inequality setting and in optimization context.

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Following the usual approach (see for instance [3]), we will assume that the function F_1, F_2 verify the following conditions:

(A1) $F(x,x)=0$ for all $x, y \in C$;
(A2) F is monotone, i.e., $F(x,y)+F(y,x) \leqslant 0$ for all $x, y \in C$;
(A3) $\limsup_{t \downarrow 0} F(tz+(1-t)x, y) \leqslant F(x,y)$ for any $x, y, z \in C$;
(A4) for each $x \in C$, $y \to F(x,y)$ is convex and lower-semicontinuous.

To begin with, let us define the domain and the graph of a bifunction F as follows

$$dom\, F = \left\{ x \in C;\ \exists v \in H \text{ with } F(x,y) + \langle v, x-y \rangle \geqslant 0,\ \forall y \in C \right\}$$

and

$$gph\, F = \left\{ (x,v) \in C \times H;\ F(x,y) + \langle v, x-y \rangle \geqslant 0,\ \forall y \in C \right\}.$$

It is worth mentioning that in the context of variational inclusions (i.e. $F(x,y) = \sup_{z \in Ax} \langle z, y-x \rangle$, A being a maximal monotone operator), $dom\, F$ is equal to $dom\, A$, and in the case of convex optimization (namely $F(x,y) = f(y) - f(x)$, f being a proper convex lower semicontinuous function), $dom\, F$ is nothing but the domain of its subdifferential ∂f. Let us recall that the maximality of a bifunction was defined, see [3], by: if for all $(x,v) \in C \times H$ with $F(y,x) + \langle v, x-y \rangle \geqslant 0$, $\forall y \in C$, then $F(x,y) + \langle v, x-y \rangle \geqslant 0$ for all $y \in C$.

Remark 1.1. It is easy to check that the monotonicity of F as a bifunction implies that of its graph, $gph\, F$, in the operator sense. On the other hand, we would like to emphasize that the maximality of a monotone bifunction F is equivalent to the maximal monotonicity of its graph, $gph\, F$, in the operator sense, see for instance [2,7,8]. Furthermore, it is worth mentioning that if F is convex with respect to the second argument and satisfies (A3), then F is maximal, see for example [4, Proposition 4.1].

Throughout, we assume that the *the solution set of the problem under consideration as well as dom $F_1 \cap$ dom F_2 are non empty*.

Now, we state two preliminary results which will be needed in the sequel.

Lemma 1.1. *(See for instance [5] or [9].) Let C be a nonempty closed convex subset of H and let F be a bi-function from $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4).

Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that:

$$F(z,y) + \frac{1}{r}\langle y-z, z-x \rangle \geqslant 0, \quad \forall y \in C.$$

Lemma 1.2. *(See [11].) Let H be a Hilbert space, (x_n) a sequence in H and set $\bar{x}_n := \frac{\sum_{i=1}^n r_i x_i}{\sum_{i=1}^n r_i}$. Let (r_n) be a sequence of nonnegative real numbers such that $\sum_{n=1}^\infty r_n = +\infty$ and assume that there exists a nonempty, closed convex set $\tilde{S} \subset H$ satisfying*:

(i) *For every $\tilde{x} \in \tilde{S}$, $\lim_n \|x_n - \tilde{x}\|$ exists.*
(ii) *Any weak-cluster point of the sequence $(\bar{x}_n)$ belongs in $\tilde{S}$.*

Then, there exists $\bar{x} \in \tilde{S}$ such that $(\bar{x}_n)$ weakly converges to $\bar{x}$.

Now, to solving (1.1), we consider the following splitting algorithm which generates, from an initial point $x_0 \in C$, three sequences (x_n), (y_n) and (z_n) by

$$\begin{cases} \text{given } x_n \text{ compute } y_n, z_n \text{ by} \\ r_n F_1(y_n, y) + \langle y_n - x_n, y - y_n \rangle \geqslant 0, \quad \forall y \in C, \\ r_n F_2(z_n, y) + \langle z_n - x_n, y - z_n \rangle \geqslant 0, \quad \forall y \in C. \\ \text{Then, compute } x_{n+1} \text{ by} \\ x_{n+1} = \dfrac{y_n + z_n}{2}, \end{cases} \tag{1.2}$$

for all $n \in \mathbb{N}$, where $(r_n) \subset (0, \infty)$

It is worth mentioning that x_n and y_n are well-defined by invoking Lemma 1.1.

2. The main convergence theorem

Now we are in a position to state our convergence result.

Theorem 2.1. *Let C be a nonempty closed convex subset of H and F_1, F_2 be two bi-functions from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4)*. Assume that (r_n) verifies $\sum_{n=1}^{\infty} r_n = +\infty$ and $\sum_{n=1}^{\infty} r_n^2 < +\infty$. Then, the sequence $\left(\bar{x}_n := \frac{\sum_{i=1}^n r_i x_i}{\sum_{i=1}^n r_i}\right)$ weakly converges to a solution of* (1.1)*.*

Proof. Let $x \in dom\, F_1 \cap dom\, F_2$, then there exists v_1, v_2 such that $(x, v_1) \in gph\, F_1$ and $(x, v_2) \in gph\, F_2$. Algorithm (1.2) and definition of $gph\, F_1$ imply

$$F_1(y_n, x) + \left\langle \frac{y_n - x_n}{r_n}, x - y_n \right\rangle \geqslant 0$$

and

$$F_1(x, y_n) + \langle v_1, x - y_n \rangle \geqslant 0.$$

By adding the last two inequalities and using monotonicity of F_1, we obtain

$$\left\langle \frac{x_n - y_n}{r_n} - v_1, y_n - x \right\rangle \geqslant 0.$$

Similarly, we have

$$\left\langle \frac{x_n - z_n}{r_n} - v_2, z_n - x \right\rangle \geqslant 0,$$

or in other words

$$\langle y_n - x_n, x - y_n \rangle \geqslant r_n \langle v_1, y_n - x \rangle,$$
$$\langle z_n - x_n, x - z_n \rangle \geqslant r_n \langle v_2, z_n - x \rangle.$$

Hence

$$\|x_n - x\|^2 - \|y_n - x\|^2 \geqslant \|x_n - y_n\|^2 + 2r_n \langle v_1, y_n - x \rangle,$$
$$\|x_n - x\|^2 - \|z_n - x\|^2 \geqslant \|x_n - z_n\|^2 + 2r_n \langle v_2, z_n - x \rangle.$$

Now, using the fact that

$$\forall y, z \in H \quad \text{one has} \quad \|y\|^2 + 2\langle y, z \rangle \geqslant -\|z\|^2,$$

we deduce

$$\|x_n - x\|^2 - \|y_n - x\|^2 \geqslant -r_n^2 \|v_1\|^2 + 2r_n \langle v_1, x_n - x \rangle, \tag{2.1}$$

and

$$\|x_n - x\|^2 - \|z_n - x\|^2 \geqslant -r_n^2 \|v_2\|^2 + 2r_n \langle v_2, x_n - x \rangle. \tag{2.2}$$

Definition of x_n together with the convexity of the norm yield

$$-\|x_{n+1} - x\|^2 \geqslant -\frac{1}{2}\|y_n - x\|^2 - \frac{1}{2}\|z_n - x\|^2.$$

Multiplying (2.1) (resp. (2.2)) by $\frac{1}{2}$ and adding the two inequalities, we obtain

$$\|x_n - x\|^2 - \|x_{n+1} - x\|^2 \geqslant -r_n^2\big(\|v_1\|^2 + \|v_2\|^2\big) + \big\langle v, r_n(x_n - x)\big\rangle, \tag{2.3}$$

where $v := v_1 + v_2$. Writing (2.3) with an index i and summing from $i = 1$ to n, we obtain

$$\frac{\|x_1 - x\|^2 - \|x_{n+1} - x\|^2}{\sum_{i=1}^n r_i} \geqslant -\frac{\sum_{i=1}^n r_i^2}{\sum_{i=1}^n r_i}\big(\|v_1\|^2 + \|v_2\|^2\big) + \langle v, \bar{x}_n - x \rangle. \tag{2.4}$$

Putting $x = x^* \in S$ in (2.3) and observing that in this case we can take $v = 0$, we obtain

$$\|x_{n+1} - x^*\|^2 \leqslant \|x_n - x^*\|^2 + 2r_n^2 \|v_1\|^2.$$

The hypothesis $\sum_{i=1}^n r_n^2 < +\infty$ secures the existence of $\lim_n \|x_n - x^*\|^2$. Hence the first assumption of Lemma 1.2 is satisfied. This implies that the sequence (x_n) is bounded and so is $(\bar{x}_n)$.

To complete the proof, let us verify that the second assumption of Lemma 2.1 holds true. First, note that F is maximal since it is convex with respect to its second argument and upper hemicontinuous with respect to its first argument. Now, let $\bar{x}$ be a weak-cluster point of $(\bar{x}_n)$, by passing to the limit in (2.4) on a subsequence, we obtain

$$\langle v, \bar{x} - x \rangle \leqslant 0, \quad \forall (x, v) \in gph\, F.$$

This implies that

$$\langle v, \bar{x} - x\rangle \leqslant 0 \quad \text{and} \quad F(x, \bar{x}) + \langle v, x - \bar{x}\rangle \geqslant 0, \quad \forall x \in C,$$

which in turn, by virtue of the maximality of F, implies that

$$\langle v, \bar{x} - x\rangle \leqslant 0 \quad \text{and} \quad F(\bar{x}, x) + \langle v, \bar{x} - x\rangle \geqslant 0, \quad \forall x \in C.$$

From which we infer that $F(\bar{x}, x) \geqslant 0$, $\forall x \in C$, or in other words $\bar{x} \in S$.

Consequently, the result follows by applying Lemma 1.2 with $\tilde{S} = S$. $\square$

Remark 2.1. It is worth mentioning that we can take a different regularization parameter, says s_n, for F_2 in (1.2). By setting $x_{n+1} = \alpha_n y_n + (1 - \alpha_n) z_n$ and ($\bar{x}_n := \frac{\sum_{i=1}^n t_i x_i}{\sum_{i=1}^n t_i}$), where $\alpha_n = \frac{s_n}{r_n + s_n}$ and $t_n := \frac{r_n s_n}{r_n + s_n} = r_n \alpha_n = (1 - \alpha_n) s_n$, and by assuming the same assumptions as in Theorem 2.1, the convergence result is still valid provided that the sequences (s_n) and (r_n) are equivalent, namely $\lim_{n \to +\infty} \frac{s_n}{r_n} = \kappa \in \mathbb{R}_+^*$.

Now, instead of computing y_n and z_n in **parallel**, another approach consists in computing them **successively**. The resulting splitting algorithm generates, from an initial point $x_0 \in C$, two sequences (x_n) and (z_n) by

$$\begin{cases} \text{given } x_{n-1} \text{ compute } z_n \text{ by} \\ r_n F_2(z_n, y) + \langle z_n - x_{n-1}, y - z_n\rangle \geqslant 0, \quad \forall y \in C. \\ \text{Then, compute } x_n \text{ by} \\ r_n F_1(x_n, y) + \langle x_n - z_n, y - x_n\rangle \geqslant 0, \quad \forall y \in C, \end{cases} \tag{2.5}$$

for all $n \in \mathbb{N}^*$, where $(r_n) \subset (0, \infty)$.

In what follows, we will show that under the hypotheses of Theorem 2.1, the sequence $\bar{x}_n := \frac{\sum_{i=1}^n r_i x_i}{\sum_{i=1}^n r_i}$ weakly converges to a solution of (1.1).

Indeed, let $x \in dom\, F_1 \cap dom\, F_2$, then there exists v_1, v_2 such that $(x, v_1) \in gph\, F_1$ and $(x, v_2) \in gph\, F_2$. Algorithm (2.5) and definition of $gph\, F_1$ imply

$$F_1(x_n, x) + \left\langle \frac{x_n - z_n}{r_n}, x - x_n \right\rangle \geqslant 0$$

and

$$F_1(x, x_n) + \langle v_1, x - x_n\rangle \geqslant 0.$$

By adding the last two inequalities and using monotonicity of F_1, we obtain

$$\left\langle \frac{z_n - x_n}{r_n} - v_1, x_n - x \right\rangle \geqslant 0.$$

Similarly, we have

$$\left\langle \frac{x_{n-1} - z_n}{r_n} - v_2, z_n - x \right\rangle \geqslant 0,$$

or in other words

$$\langle z_n - x_n, x_n - x\rangle \geqslant r_n \langle v_1, x_n - x\rangle \tag{2.6}$$

and

$$\langle x_{n-1} - z_n, z_n - x\rangle \geqslant r_n \langle v_2, z_n - x\rangle. \tag{2.7}$$

From (2.6) and (2.7), using the general equality

$$2\langle a - b, b - c\rangle = \|a - c\|^2 - \|a - b\|^2 - \|b - c\|^2,$$

we obtain

$$\|x_{n-1} - x\|^2 - \|z_n - x\|^2 \geqslant \|x_{n-1} - z_n\|^2 + 2r_n \langle v_2, z_n - x\rangle,$$

and

$$\|z_n - x\|^2 - \|x_n - x\|^2 \geqslant \|z_n - x_n\|^2 + 2r_n \langle v_1, x_n - x\rangle.$$

By adding the two inequalities, we obtain

$$\begin{aligned}\|x_{n-1}-x\|^2-\|x_n-x\|^2 &\geqslant \|x_{n-1}-z_n\|^2+\|z_n-x_n\|^2+2r_n\langle v_2, z_n-x\rangle+2r_n\langle v_1, x_n-x\rangle\\ &\geqslant \|z_n-x_n\|^2+2r_n\langle v_2, z_n-x\rangle+2r_n\langle v_1, x_n-x\rangle\\ &= \|z_n-x_n\|^2+2r_n\langle v_2, z_n-x\rangle-2r_n\langle v_2, x_n-x\rangle+2r_n\langle v, x_n-x\rangle\\ &= \|z_n-x_n\|^2+2r_n\langle v_2, z_n-x_n\rangle+2r_n\langle v, x_n-x\rangle.\end{aligned}$$

Therefore,

$$\|x_{n-1}-x\|^2-\|x_n-x\|^2 \geqslant -r_n^2\|v_2\|^2+2r_n\langle v, x_n-x\rangle, \tag{2.8}$$

where $v := v_1+v_2$. The end of the proof follows that of Theorem 2.1.

Remark 2.2. (2.5) relies on a regularization of one of the bifunctions. Actually, replacing (1.1) by

$$\text{find } x\in C \quad \text{such that} \quad F_1(x,y)+\langle A_r^{F_2}(x), y-x\rangle \geqslant 0, \quad \forall y\in C, \tag{2.9}$$

where $A_r^{F_2}(x) := \frac{x-J_r^{F_2}(x)}{r}$ stands for the Yosida operator introduced by Moudafi and Théra in [9] and $J_r^{F_2}(x)$ denotes the unique solution z defined in Lemma 1.1.

According to Lemma 1.1, we easily deduce that x solves (2.9) if and only if x is a solution to the following fixed point problem

$$\text{find } x\in C \quad \text{such that} \quad x = J_r^{F_1}\circ J_r^{F_2}(x).$$

(2.5) is obtained by iterating the above relation with a variable parameter r_n.

The barycentric–proximal method stated in Remark 2.1 is based on a complete regularization of the two bifunctions under consideration. This consists in replacing (1.1) by the regularized problem

$$\text{find } x\in C \quad \text{such that} \quad \langle A_r^{F_1}(x), y-x\rangle+\langle A_s^{F_2}(x), y-x\rangle \geqslant 0, \quad \forall y\in C.$$

By taking $y=(1-(\frac{1}{r}+\frac{1}{s}))x+\frac{1}{r}J_r^{F_1}(x)+\frac{1}{s}J_s^{F_2}(x)\in C$, we obtain that x is a solution of the following fixed-point problem

$$\text{find } x\in C \quad \text{such that} \quad x=\frac{s}{r+s}J_r^{F_1}(x)+\frac{r}{r+s}J_s^{F_2}(x). \tag{2.10}$$

The barycentric–proximal method is nothing but the iteration method of (2.10) with variable parameters r_n and s_n. For sake of simplicity it is taken $r_n=s_n$ for all $n\in\mathbb{N}$ to obtain (2.1).

3. Applications

As a direct consequence of Theorem 2.1, we obtain the following:

1. Constrained convex programming: By taking $F_i(x,y)=\varphi_i(y)-\varphi_i(x)$, $i=1,2$ where φ_i is a proper convex, lower semi-continuous function. S is nothing but $\mathit{argmin}_C(\varphi_1+\varphi_2)$ and (1.2) (resp. (2.5)) reduces to

$$x_{n+1}=\frac{\mathit{prox}_{r_n\varphi_1}(x_n)+\mathit{prox}_{r_n\varphi_2}(x_n)}{2} \quad \big(\text{resp. } x_n=\mathit{prox}_{r_n\varphi_1}\big(\mathit{prox}_{r_n\varphi_2}(x_{n-1})\big)\big),$$

where $\mathit{prox}_{r_n\varphi_i}(x_n)=\arg\min_C\{\varphi_i(x)+\frac{1}{2r_n}\|x-x_n\|^2\}$. As a corollary of Theorem 2.1 we obtain the weak ergodic convergence of (x_n) to a minimizer of $\varphi_1+\varphi_2$ on C.

2. Fixed-point problem: We can obtain an ergodic convergence result for a pair of pseudo-nonexpansive mappings by setting $F_i(x,y)=\langle x-P_ix, y-x\rangle$, $i=1,2$ where $P_i: C\to C$ is a nonlinear mapping. It is worth mentioning that in this case, the monotonicity condition on F is equivalent to saying that the mapping P_i is pseudo-nonexpansive, that is $\langle P_ix-P_iy, x-y\rangle \leqslant \|x-y\|^2$. In this context, S is equal to $\mathit{Fix}(\frac{1}{2}(P+Q))$ and we obtain the ergodic convergence of (x_n) generated by (1.2) (resp. (2.5)), which reduces to the following implicit method: $x_{n+1}=\frac{y_n+z_n}{2}$ where y_n, z_n are defined implicitly by

$$y_n=\frac{1}{1+r_n}x_n+\frac{r_n}{1+r_n}P_1(y_n) \quad \text{and} \quad z_n=\frac{1}{1+r_n}x_n+\frac{r_n}{1+r_n}P_2(z_n),$$
$$x_n=\frac{1}{1+r_n}z_n+\frac{r_n}{1+r_n}P_1(x_n) \quad \text{and} \quad z_n=\frac{1}{1+r_n}x_{n-1}+\frac{r_n}{1+r_n}P_2(z_n),$$

respectively. It is well known that $\mathit{Fix}(\frac{1}{2}(P_1+P_2))$ is equal to $\mathit{Fix}(P_1)\cap\mathit{Fix}(P_2)$ where the latter is nonempty and the mappings P_1, P_2 are nonexpansive.

3. Set-valued operators: Now, by taking $C = H$, $F_i(x, y) = \sup_{\xi \in T_i(x)} \langle \xi, y - x \rangle$ such that $T_1 + T_2$ is a maximal monotone operator, where T_i is a maximal monotone operator, (1.2) (resp. (2.5)) reduces to the following method

$$x_{n+1} = \frac{(I + r_n T_1)^{-1}(x_n) + (I + r_n T_2)^{-1}(x_n)}{2},$$
$$x_n = (I + r_n T_1)^{-1}\big((I + r_n T_2)^{-1}(x_{n-1})\big),$$

respectively, and we obtain as a particular case a result in [6] (resp. in [11]).

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It is a pleasure and honor to dedicate this work to Professor Wataru Takahashi on the occasion of his retirement.

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